

Non-reduced components of the Noether-Lefschetz locus

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Abstract

In this article we give a criterion for a component of the Noether-Lefschetz locus to be non-reduced. We also produce several examples of such components.

1 Introduction

In 1882, M. Noether claimed the following statement which was later proven by Lefschetz: For $d \geq 4$, a very general smooth degree d surface X in \mathbb{P}^3 has Picard number, denoted $\rho(X)$ equal to 1. This motivates the definition of the *Noether-Lefschetz locus*, denoted by NL_d parametrizing the space of smooth degree d surfaces X in \mathbb{P}^3 with $\rho(X) > 1$. One of the interesting problems is the geometry of the Noether-Lefschetz locus. We know that an irreducible component L of NL_d is an algebraic scheme [CDK95] and is not necessarily reduced. In this article we study the scheme structure of L and give a necessary and sufficient condition for non-reducedness of L . Finally, we give examples of such components.

One of the first results in this direction is due to Green, Griffiths, Voisin and others ([Gre89, GH83, Voi88]) which states that for an irreducible component L of the Noether-Lefschetz locus and $d \geq 4$, the codimension of L in U_d satisfies the following inequality:

$$d - 3 \leq \text{codim}(L, U_d) \leq \binom{d-1}{3}.$$

The upper bound follows easily from the fact that $\dim H^{2,0}(X) = \binom{d-1}{3}$ for any $X \in U_d$ (see [Voi03, §6]). We say that L is a *general* component if $\text{codim } L = \binom{d-1}{3}$ and *special* otherwise.

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It was proven by Ciliberto, Harris and Miranda [CHM88] that for $d \geq 4$, the Noether-Lefschetz locus has infinitely many general components and the union of these components is Zariski dense in U_d . The guiding principle of much work in the area has been the expectation that special components should be due to the presence of low degree curves. Voisin [Voi89] and Green [Gre89] independently proved that for $d \geq 5$, $\text{codim } L = d - 3$ if and only if L parametrizes surfaces of degree d containing a line. This component is reduced. If $d - 3 < \text{codim } L \leq 2d - 7$ then $\text{codim } L = 2d - 7$ and L parametrizes the surfaces containing a conic and is reduced. Maclean [Mac05] showed that for $d = 5$ there exists a component L of codimension $2d - 6$ such that L_{red} , the associated reduced subscheme, parametrizes the surfaces containing two lines on the same plane and this component is non-reduced. Otwinowska [Otw03] proved that for an integer $b > 0$ and $d \gg b$ if $\text{codim } L \leq bd$ then L parametrizes surfaces containing a curve of degree at most b .

We now briefly discuss the main ideas used in this article. One of the important observations is that an irreducible component of the Noether-Lefschetz locus is a Hodge locus. This is a consequence of the Lefschetz $(1, 1)$ -theorem. We recall the notion of the Hodge locus. Denote by $U_d \subseteq \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)))$ the open subscheme parametrizing smooth projective hypersurfaces in \mathbb{P}^3 of degree d . Let $\mathcal{X} \xrightarrow{\pi} U_d$ be the corresponding universal family. For a given $F \in U_d$, denote by X_F the surface $X_F := \pi^{-1}(F)$. Let $X \in U_d$ and $U \subseteq U_d$ be a simply connected neighbourhood of X in U_d (under the analytic topology). Then $\pi|_{\pi^{-1}(U)}$ induces a variation of Hodge structure (\mathcal{H}, ∇) on U where $\mathcal{H} := R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U$ and ∇ is the Gauss-Manin connection. Note that \mathcal{H} defines a local system on U whose fiber over a point $F \in U$ is $H^2(X_F, \mathbb{Z})$ where $X_F = \pi^{-1}(F)$. Consider a non-zero element $\gamma_0 \in H^2(X_F, \mathbb{Z}) \cap H^{1,1}(X_F, \mathbb{C})$ such that $\gamma_0 \neq c_1(\mathcal{O}_{X_F}(k))$ for $k \in \mathbb{Z}_{>0}$. This defines a section $\gamma \in (\mathcal{H} \otimes \mathbb{C})(U)$. Let $\overline{\gamma}$ be the image of γ in $\mathcal{H}/F^2(\mathcal{H} \otimes \mathbb{C})$. The Hodge loci, denoted $\text{NL}(\gamma)$ is then defined as

$$\text{NL}(\gamma) := \{G \in U | \overline{\gamma}_G = 0\},$$

where $\overline{\gamma}_G$ denotes the value at G of the section $\overline{\gamma}$. For an irreducible component $L \subset \text{NL}_d$ and $X \in L$, general, we can find $\gamma \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})$ such that $\overline{\text{NL}(\gamma)} = L$ (the closure taken under Zariski topology on U_d). The Gauss-Manin connection gives rise to a differential map, $\overline{\nabla}(\gamma) : T_X U \rightarrow H^2(\mathcal{O}_X)$, where $T_X U$ is the tangent space to U at the point corresponding to X .

The tangent space at X to $\text{NL}(\gamma)$ is defined to be $\ker(\overline{\nabla}(\gamma))$.

For a better understanding of the tangent space to $\text{NL}(\gamma)$ we will use the theory of semi-regularity as introduced by Bloch [Blo72]. Recall, given a curve C in a smooth surface X in \mathbb{P}^3 , the semi-regularity map is the morphism $\pi : H^1(\mathcal{N}_{C|X}) \rightarrow H^2(\mathcal{O}_X)$ arising from the short exact sequence,

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{N}_{C|X} \rightarrow 0.$$

We show in Theorem 4.8:

Theorem 1.1. Let X be a smooth surface in \mathbb{P}^3 , C a curve in X and γ the cohomology class of C . Then, the differential map $\overline{\nabla}(\gamma) : H^0(\mathcal{N}_{X|\mathbb{P}^3}) \rightarrow H^2(\mathcal{O}_X)$ factors through the semi-regularity map.

Using this we can compare infinitesimal deformations of the pair (C, X) such that C remains a curve to the infinitesimal deformations of X such that the cohomology class of C remains a Hodge class.

We then use this to give a necessary and sufficient condition for non-reducedness of an irreducible component of NL_d :

Theorem 1.2. Consider an irreducible component L of the Noether-Lefschetz locus. Then, for a general element $X \in L$, there exists $\gamma \in H^{1,1}(X, \mathbb{Z})$ satisfying the following conditions:

1. γ is the cohomology of a curve, say C in X , such that the corresponding semi-regularity map is injective (i.e., C is semi-regular)
2. There is an irreducible component, say H_γ of the flag Hilbert scheme parametrizing pairs $(C' \subset X')$ for C' a curve (resp. X' a surface) with Hilbert polynomial the same as C (resp. X) such that $\text{pr}_2(H_\gamma)_{\text{red}}$ is isomorphic to L_{red} and $\text{pr}_2(T_{(C,X)}H_\gamma)$ is the same as $T_X(\text{NL}(\gamma))$.
3. $H^1(\mathcal{O}_X(-C)(d)) = 0$ i.e., for every infinitesimal deformation of C there exists a corresponding infinitesimal deformation of X containing it.

Furthermore, L is non-reduced if and only if $\dim \text{pr}_1(H_\gamma) < h^0(\mathcal{N}_{C|\mathbb{P}^3})$.

See Proposition 5.8, Corollary 5.9 and Theorem 5.13 for the precise statements and proofs. The proofs of the parts (1) and (3) is an application of Serre's vanishing theorem. Part (2) uses the description of $\overline{\nabla}(\gamma)$ in terms of the semi-regularity map as given in Theorem 4.8.

In order to give examples in the case L is non-reduced, we fix an integer $d \geq 5$ and consider certain irreducible non-reduced components of the Hilbert scheme of curves in \mathbb{P}^3 such that the general element C is contained in a smooth degree d surface, X . We show in Lemma 5.2 and Theorem 6.11 that if $d - 4 \geq \deg(C)$ then $\mathcal{O}_X(-C)$ is d -regular in the sense of Castelnuovo-Mumford. This tells us that given an infinitesimal deformation of such C there exists a corresponding infinitesimal deformation of X containing it. Using this we show that the Hodge locus corresponding to the cohomology class of C is non-reduced. Among other examples we show that,

Theorem 1.3. Let $d \geq 5$, X a surface containing two distinct coplanar lines, say l_1, l_2 , C a divisor in X of the form $2l_1 + l_2$ and γ the cohomology class of C . Then, $\overline{\text{NL}}(\gamma)$ (closure taken in U_d under Zariski topology) is non-reduced.

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2 Introduction to Noether-Lefschetz locus

2.1. In this section we recall the basic definitions of Noether-Lefschetz locus. See [Voi02, §9, 10] and [Voi03, §5, 6] for a detailed presentation of the subject.

Definition 2.2. Recall, for a fixed integer $d \geq 5$, the *Noether-Lefschetz locus*, denoted NL_d , parametrizes the space of smooth degree d surfaces in \mathbb{P}^3 with Picard number greater than 1. Using the Lefschetz $(1, 1)$ -theorem this is the parametrizing space of smooth degree d surfaces with $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}) \neq \mathbb{Z}$.

Notation 2.3. Let $X \in U_d$ and $\mathcal{O}_X(1)$, the very ample line bundle on X determined by the closed immersion $X \hookrightarrow \mathbb{P}^3$ arising (as in [Har77, II.Ex.2.14(b)]) from the graded homomorphism

of graded rings $S \rightarrow S/(F_X)$, where $S = \Gamma_*(\mathcal{O}_{\mathbb{P}^3})$ and F_X is the defining equations of X . Denote by H_X the very ample line bundle $\mathcal{O}_X(1)$.

Notation 2.4. Let X be a surface. Denote by $H^2(X, \mathbb{C})_{\text{prim}}$, the primitive cohomology. There is a natural projection map from $H^2(X, \mathbb{C})$ to $H^2(X, \mathbb{C})_{\text{prim}}$. For $\gamma \in H^2(X, \mathbb{C})$, denote by γ_{prim} the image of γ under this morphism. Since the very ample line bundle H_X remains of type $(1, 1)$ in the family \mathcal{X} , we can conclude that $\gamma \in H^{1,1}(X)$ remains of type $(1, 1)$ if and only if γ_{prim} remains of type $(1, 1)$. In particular, $\text{NL}(\gamma) = \text{NL}(\gamma_{\text{prim}})$.

2.5. Note that, NL_d is a countable union of subvarieties. Every irreducible component of NL_d is locally of the form $\text{NL}(\gamma)$ for some $\gamma \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$, $X \in \text{NL}_d$ such that $\gamma_{\text{prim}} \neq 0$.

Lemma 2.6 ([Voi03, Lemma 5.13]). There is a natural analytic scheme structure on $\overline{\text{NL}(\gamma)}$ (closure in U_d under Zariski topology).

Notation 2.7. Let X_1 be a projective scheme, $X_2 \subset X_1$, a closed subscheme. Denote by $\mathcal{N}_{X_2|X_1}$ the normal sheaf $\text{Hom}_{X_1}(\mathcal{I}_{X_2/X_1}, \mathcal{O}_{X_2})$, where \mathcal{I}_{X_2/X_1} is the ideal sheaf of X_2 in X_1 .

Definition 2.8. We now discuss the tangent space to the Hodge locus, $\text{NL}(\gamma)$. We know that the tangent space to U at X , $T_X U$ is isomorphic to $H^0(\mathcal{N}_{X|\mathbb{P}^3})$. This is because U is an open subscheme of the Hilbert scheme H_{Q_d} , the tangent space of which at the point X is simply $H^0(\mathcal{N}_{X|\mathbb{P}^3})$. Given the variation of Hodge structure above, we have (by Griffith's transversality) the differential map:

$$\overline{\nabla} : H^{1,1}(X) \rightarrow \text{Hom}(T_X U, H^2(X, \mathcal{O}_X))$$

induced by the Gauss-Manin connection. Given $\gamma \in H^{1,1}(X)$ this induces a morphism, denoted $\overline{\nabla}(\gamma)$ from $T_X U$ to $H^2(\mathcal{O}_X)$.

Lemma 2.9 ([Voi03, Lemma 5.16]). The tangent space at X to $\text{NL}(\gamma)$ equals $\ker(\overline{\nabla}(\gamma))$.

Definition 2.10. The boundary map ρ , from $H^0(\mathcal{N}_{X|\mathbb{P}^3})$ to $H^1(\mathcal{T}_X)$ arising from the long exact sequence associated to the short exact sequence:

$$0 \rightarrow \mathcal{T}_X \rightarrow \mathcal{T}_{\mathbb{P}^3}|_X \rightarrow \mathcal{N}_{X|\mathbb{P}^3} \rightarrow 0 \tag{1}$$

is called the *Kodaira-Spencer* map. The morphism $\overline{\nabla}(\gamma)$ is related to the Kodaira-Spencer map as follows:

Theorem 2.11. The differential map $\overline{\nabla}(\gamma)$ coincides with the following:

$$T_X U \cong H^0(\mathcal{N}_{X|\mathbb{P}^3}) \xrightarrow{\rho} H^1(\mathcal{T}_X) \xrightarrow{\bigcup_{\gamma}} H^2(\mathcal{O}_X)$$

and under the identification $\mathcal{N}_{X|\mathbb{P}^3} \cong \mathcal{O}_X(d)$, $\ker(\rho) \cong J_d(F)$, where $J_d(F)$ denotes the degree d graded piece of the Jacobian ideal of X .

Proof. See [Voi02, Theorem 10.21] and [Voi03, Lemma 6.15] for a proof. \square

3 Introduction to flag Hilbert schemes

We briefly recall the basic definition of flag Hilbert schemes and its tangent space in the relevant case. See [Ser06, §4.5] for further details.

Definition 3.1. Given an m -tuple of numerical polynomials $\mathcal{P}(t) = (P_1(t), P_2(t), \dots, P_m(t))$, we define the contravariant functor, called the *Hilbert flag functor* relative to $\mathcal{P}(t)$,

$$FH_{\mathcal{P}(t)} : (\text{schemes}) \rightarrow \text{sets}$$

$$S \mapsto \left\{ (X_1, X_2, \dots, X_m) \left| \begin{array}{l} X_1 \subset X_2 \subset \dots \subset X_m \subset \mathbb{P}_S^3 \\ X_i \text{ are } S\text{-flat with Hilbert polynomial } P_i(t) \end{array} \right. \right\}$$

We call such an m -tuple a *flag relative to $\mathcal{P}(t)$* . The functor is representable by a projective scheme $H_{\mathcal{P}(t)}$, called *flag Hilbert scheme*.

Definition 3.2. In the case $m = 2$, we have the following definition of the tangent space at a pair (X_1, X_2) to the flag Hilbert scheme H_{P_1, P_2} :

$$\begin{array}{ccc} T_{(X_1, X_2)} H_{P_1, P_2} & \longrightarrow & H^0(\mathcal{N}_{X_2|\mathbb{P}^3}) \\ \downarrow & \square & \downarrow \\ H^0(\mathcal{N}_{X_1|\mathbb{P}^3}) & \longrightarrow & H^0(\mathcal{N}_{X_2|\mathbb{P}^3} \otimes \mathcal{O}_{X_1}) \end{array} \quad (2)$$

Definition 3.3. We now fix certain notation for the rest of this article. By a *component* of NL_d , we mean an irreducible component. By a *surface* we always mean a smooth surface in \mathbb{P}^3 . Denote by Q_d the Hilbert polynomial of degree d surfaces in \mathbb{P}^3 . Given, a Hilbert polynomial P , denote by H_P the corresponding Hilbert scheme and by H_{P,Q_d} the corresponding flag Hilbert scheme.

4 Hodge locus and semi-regularity

4.1. In this section, we consider the case when γ is the cohomology class of a curve C in a smooth degree d surface X in \mathbb{P}^3 . We see that the differential map $\overline{\nabla}(\gamma)$ factors through the semi-regularity map, $H^1(\mathcal{N}_{C|X}) \rightarrow H^2(\mathcal{O}_X)$. Finally, using this description, we compute $\ker \overline{\nabla}(\gamma)$, which is the tangent space to the Hodge locus, $\text{NL}(\gamma)$.

4.2. We start with the definition of a semi-regular curve. Let X be a surface and $C \subset X$, a curve in X . Since X is smooth, C is local complete intersection in X . Denote by i the closed immersion of C into X . This gives rise to the short exact sequence:

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{N}_{C|X} \rightarrow 0. \quad (3)$$

The *semi-regularity map* π is the boundary map from $H^1(\mathcal{N}_{C|X})$ to $H^2(\mathcal{O}_X)$. We say that C is *semi-regular* if π is injective.

4.3. Let X be a smooth surface and C a local complete intersection curve in X . Consider the natural differential map, $d_X : \mathcal{O}_X(-C) \otimes \mathcal{O}_C \rightarrow \Omega_X^1 \otimes \mathcal{O}_C$. This yields a map $\text{Ext}_C^1(\Omega_X^1 \otimes \mathcal{O}_C, \mathcal{O}_C) \rightarrow \text{Ext}_C^1(\mathcal{O}_X(-C) \otimes \mathcal{O}_C, \mathcal{O}_C)$. Since $\Omega_X^1 \otimes \mathcal{O}_C$ and $\mathcal{O}_X(-C) \otimes \mathcal{O}_C$ are locally free \mathcal{O}_C -modules, [Har77, III.Ex. 6.5] implies that

$$\mathcal{E}xt_C^1(\Omega_X^1 \otimes \mathcal{O}_C, \mathcal{O}_C) = 0 = \mathcal{E}xt_C^1(\mathcal{O}_X(-C) \otimes \mathcal{O}_C, \mathcal{O}_C).$$

Using the local to global Ext spectral sequence, we can conclude that $\text{Ext}_C^1(\Omega_X^1 \otimes \mathcal{O}_C, \mathcal{O}_C)$ (resp. $\text{Ext}_C^1(\mathcal{O}_X(-C) \otimes \mathcal{O}_C, \mathcal{O}_C)$) is isomorphic to $H^1(\mathcal{T}_X \otimes \mathcal{O}_C)$ (resp. $H^1(\mathcal{N}_{C|X})$) where \mathcal{T}_X is the tangent sheaf on X and $\mathcal{N}_{C|X}$ is the normal sheaf of C in X . Let $u_* : H^1(\mathcal{T}_X) \rightarrow H^1(\mathcal{N}_{C|X})$ be

the composition of the restriction morphism $j_3 : H^1(\mathcal{T}_X) \rightarrow H^1(\mathcal{T}_X \otimes \mathcal{O}_C)$ with the morphism $j_1 : H^1(\mathcal{T}_X \otimes \mathcal{O}_C) \rightarrow H^1(\mathcal{N}_{C|X})$ obtained above.

Theorem 4.4 ([BF00, Theorem 4.5, 5.5]). Let C, X be as in 4.3 and $[C]$ denote the cohomology class of C . The morphism u_* defined above satisfies the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathcal{T}_X) & \xrightarrow{u_*} & H^1(\mathcal{N}_{C|X}) \\ \cup[C] \downarrow & \nearrow \cap & \\ H^2(\mathcal{O}_X) & & \end{array}$$

where $\cup[C]$ is the morphism which takes ξ to $\xi \cup [C]$, the cup-product of ξ and $[C]$.

Remark 4.5. One can see that u_* is the obstruction map in the sense, for $\xi \in H^1(\mathcal{T}_X)$, $u_*(\xi) = 0$ if and only if C lifts to a local complete intersection in the infinitesimal deformation of ξ (see [Blo72, Proposition 2.6]). We will now see a diagram (in Theorem 4.8) which illustrates this fact more clearly.

4.6. Recall, the following short exact sequence of normal sheaves:

$$0 \rightarrow \mathcal{N}_{C|X} \rightarrow \mathcal{N}_{C|\mathbb{P}^3} \rightarrow \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C \rightarrow 0 \quad (4)$$

which arises from the short exact sequence:

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_C \xrightarrow{j^\#} j_* \mathcal{O}_X(-C) \rightarrow 0. \quad (5)$$

after applying $\mathcal{H}om_{\mathbb{P}^3}(-, j_{0*} \mathcal{O}_C)$, where j_0 is the closed immersion of C into \mathbb{P}^3 .

The following lemma will be used in the proof of Theorem 4.8 below.

Lemma 4.7. Given $j : C \hookrightarrow X$ the closed immersion, the following diagrams are commutative

and the horizontal rows are exact:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{I}_X \otimes \mathcal{O}_C & \longrightarrow & \mathcal{I}_C \otimes \mathcal{O}_C & \xrightarrow{u} & \mathcal{O}_X(-C) \otimes \mathcal{O}_C \longrightarrow 0 \\
& & \downarrow \cong & & \downarrow \circlearrowleft & & \downarrow \circlearrowleft \\
& & \mathcal{I}_X \otimes \mathcal{O}_C & \xrightarrow{d_{\mathbb{P}^3}} & \Omega_{\mathbb{P}^3}^1 \otimes \mathcal{O}_C & \xrightarrow{v} & \Omega_X^1 \otimes \mathcal{O}_C \longrightarrow 0 \\
& & & & \downarrow d_X & & \\
& & & & \Omega_X^1 \otimes \mathcal{O}_C & &
\end{array}$$

where $d_{\mathbb{P}^3}$ (resp. d_X) are the Kähler differential operator, u is induced by $j^\#$ and v maps $d_{\mathbb{P}^3} f \otimes g$ to $gd_X j^\#(f)$ on open sets.

Proof. The exactness of the lower horizontal sequence is explained in [Mat89, Theorem 25.2]. We now show the exactness of the upper horizontal sequence. The sequence is obtained via pulling back (5) by the closed immersion of C into \mathbb{P}^3 . Since tensor product is right exact, it remains to prove that the morphism from $\mathcal{I}_X \otimes \mathcal{O}_C$ to $\mathcal{I}_C \otimes \mathcal{O}_C$ is injective. It suffices to prove this statement on the stalk for all $x \in C$, closed point. Note that, $\mathcal{I}_{X,x} \otimes \mathcal{O}_{C,x}$ (resp. $\mathcal{I}_{C,x} \otimes \mathcal{O}_{C,x}$) is isomorphic to $\mathcal{I}_{X,x} / \mathcal{I}_{X,x} \mathcal{I}_{C,x}$ (resp. $\mathcal{I}_{C,x} / \mathcal{I}_{C,x}^2$). So, we need to show that $\mathcal{I}_{X,x} \cap \mathcal{I}_{C,x}^2$ is contained in $\mathcal{I}_{X,x} \cdot \mathcal{I}_{C,x}$.

Since C is a local complete intersection curve, $\mathcal{I}_{C,x}$ is generated by a regular sequence, say (f_x, g_x) where g_x generates the ideal $\mathcal{I}_{X,x}$. So, $\mathcal{I}_{X,x} \cdot \mathcal{I}_{C,x} = (g_x^2, f_x g_x)$, $\mathcal{I}_{C,x}^2 = (f_x^2, f_x g_x, g_x^2)$. Note that any element in $(g_x) \cap (f_x^2, f_x g_x, g_x^2)$ is divisible by f_x^2 modulo the ideal $(g_x^2, f_x g_x)$. Therefore it is divisible by $g_x f_x^2$, hence is an element in $\mathcal{I}_{X,x} \cdot \mathcal{I}_{C,x}$. It directly follows that the natural morphism from $\mathcal{I}_{X,x} / \mathcal{I}_{X,x} \cdot \mathcal{I}_{C,x}$ to $\mathcal{I}_{C,x} / \mathcal{I}_{C,x}^2$ is injective.

The commutativity of the diagrams follows directly from the description of the relevant morphisms. \square

Theorem 4.8. Let X be a smooth surface, $C \subset X$ and $\gamma = [C] \in H^{1,1}(X, \mathbb{Z})$. We then have the following commutative diagram

$$\begin{array}{ccccccc}
& & T_{(C,X)} H_{P,Q_d} & \longrightarrow & H^0(X, \mathcal{N}_{X|\mathbb{P}^3}) & \xrightarrow{\overline{\nabla}(\gamma)} & H^2(X, \mathcal{O}_X) \\
& & \downarrow & & \downarrow \rho_C & & \uparrow \pi_C \\
0 & \longrightarrow & H^0(C, \mathcal{N}_{C|X}) & \xrightarrow{\phi_C} & H^0(C, \mathcal{N}_{C|\mathbb{P}^3}) & \xrightarrow{\beta_C} & H^0(C, \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C) \xrightarrow{\delta_C} H^1(C, \mathcal{N}_{C|X})
\end{array}$$

where the horizontal exact sequence comes from (4), π_C is the semi-regularity map and ρ_C is the natural pull-back morphism.

Proof. The only thing left to prove is that $\overline{\nabla}(\gamma)$ is the same as $\pi_C \circ \delta_C \circ \rho_C$. Using Theorems 2.11 and 4.4, we have that $\overline{\nabla}(\gamma)$ factors as:

$$H^0(\mathcal{N}_{X|\mathbb{P}^3}) \xrightarrow{\rho} H^1(\mathcal{T}_X) \xrightarrow{u_*} H^1(\mathcal{N}_{C|X}) \xrightarrow{\pi_C} H^1(\mathcal{O}_X).$$

Hence it suffices to show that $u_* \circ \rho$ is the same as $\delta_C \circ \rho_C$.

Recall, under the notations as in 4.3 we can factor u_* as $j_1 \circ j_3$. Hence, it suffices to construct a morphism $j_2 : H^0(\mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C) \rightarrow H^1(\mathcal{T}_X \otimes \mathcal{O}_C)$ such that following two diagrams commute:

$$\begin{array}{ccccccc} H^0(\mathcal{N}_{X|\mathbb{P}^3}) & \xrightarrow{\rho} & H^1(\mathcal{T}_X) & \xrightarrow{j_3} & H^1(\mathcal{T}_X \otimes \mathcal{O}_C) & \xrightarrow{j_1} & H^1(\mathcal{N}_{C|X}) \\ & \searrow \rho_C & & \nearrow j_2 & \nearrow \delta_C & & \\ & & H^0(\mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C) & & & & \end{array} \quad (6)$$

We define j_2 in the following way: Take the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_X & \longrightarrow & \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_X & \longrightarrow & \mathcal{N}_{X|\mathbb{P}^3} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}_X \otimes \mathcal{O}_C & \longrightarrow & \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_C & \longrightarrow & \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C & \longrightarrow & 0 \end{array} \quad (7)$$

Then j_2 arises from the associated long exact sequence:

$$\begin{array}{ccc} H^0(\mathcal{N}_{X|\mathbb{P}^3}) & \xrightarrow{\rho} & H^1(X, \mathcal{T}_X) \\ \downarrow \rho_C & \circlearrowleft & \downarrow j_3 \\ H^0(\mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C) & \xrightarrow{j_2} & H^1(\mathcal{T}_X \otimes \mathcal{O}_C) \end{array}$$

where the maps (other than j_2) is the same as defined above. This gives the commutativity of the first square in the diagram (6).

We now prove the commutativity of the second diagram. Since the terms in the short exact

sequences in Lemma 4.7 are locally free \mathcal{O}_C -modules, we get the dual diagram of short exact sequence by applying $\mathcal{H}om_C(-, \mathcal{O}_C)$ to it. This gives us the following:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N}_{C|X} & \longrightarrow & \mathcal{N}_{C|\mathbb{P}^3} & \longrightarrow & \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & & \circlearrowleft & & \circlearrowleft & \cong & & \\
0 & \longrightarrow & \mathcal{T}_X \otimes \mathcal{O}_C & \longrightarrow & \mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}_C & \longrightarrow & \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C & \longrightarrow & 0
\end{array}$$

where the bottom short exact sequence is the same as in (7). Taking the associated long exact sequence, we get

$$\begin{array}{ccc}
& H^0(\mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C) & \\
\swarrow j_2 & \downarrow \delta_C & \\
H^1(\mathcal{T}_X \otimes \mathcal{O}_C) & \xrightarrow{j_1} & H^1(\mathcal{N}_{C|X})
\end{array}$$

The theorem follows. \square

Corollary 4.9. Let X be a smooth degree d surface in \mathbb{P}^3 , $C \subset X$ be a curve with Hilbert polynomial, say P and $[C]$ its corresponding cohomology class. Then, the tangent space,

$$T_X(\text{NL}([C])) \subset \rho_C^{-1}(\text{Im } \beta_C) = \text{pr}_2 T_{(C,X)} H_{P,Q_d}.$$

Furthermore, if C is semi-regular then we have equality $T_X(\text{NL}([C])) = \text{pr}_2 T_{(C,X)} H_{P,Q_d}$.

Proof. The first statement follows directly from the diagram in Theorem 4.8. Now, if C is semi-regular then π_C (as in the diagram) is injective. Hence, $\ker \overline{\nabla}([C]) = \ker(\delta_C \circ \rho_C) = \rho_C^{-1}(\text{Im } \beta_C)$. The corollary then follows. \square

Corollary 4.10. Notations as in Theorem 4.8. The kernel of ρ_C is isomorphic to $H^0(\mathcal{O}_X(-C)(d))$ and ρ_C is surjective if and only if $H^1(\mathcal{O}_X(-C)(d)) = 0$. Moreover, if $H^1(\mathcal{O}_X(-C)(d)) = 0$ then $\text{pr}_1(T_{(C,X)} H_{P,Q_d}) = H^0(\mathcal{N}_{C|\mathbb{P}^3})$.

Proof. Since $\mathcal{N}_{X|\mathbb{P}^3} \cong \mathcal{O}_X(d)$ the first statement follows from the short exact sequence,

$$0 \rightarrow \mathcal{O}_X(-C)(d) \rightarrow \mathcal{O}_X(d) \rightarrow i_* \mathcal{O}_C(d) \rightarrow 0 \quad (8)$$

and the fact that $H^1(\mathcal{O}_X(d)) = 0$, where i is the closed immersion of C into X .

The last statement follows directly from the definition of $T_{(C,X)}H_{P,Q_d}$ given in 3.2. \square

5 Criterion for Non-reducedness

5.1. In this section we demonstrate the relation between Hodge locus and certain flag Hilbert schemes. Using this we give a criterion for non-reducedness of components of the Noether-Lefschetz locus.

We first prove a result that would help us determine when a curve is semi-regular.

Lemma 5.2. Let C be a connected reduced curve and X a smooth degree d surface containing C . Then, $H^1(\mathcal{O}_X(-C)(k)) = 0$ for all $k \geq \deg(C)$. In particular, if $d \geq \deg(C) + 4$ then $h^1(\mathcal{O}_X(C)) = 0$, hence C is semi-regular.

Proof. Since X is a hypersurface in \mathbb{P}^3 of degree d , $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-d)$. Consider the short exact sequence:

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_X(-C) \rightarrow 0.$$

We get the following terms in the associated long exact sequence:

$$\dots \rightarrow H^1(\mathcal{I}_C(k)) \rightarrow H^1(\mathcal{O}_X(-C)(k)) \rightarrow H^2(\mathcal{I}_X(k)) \rightarrow \dots$$

Now, $H^2(\mathcal{O}_{\mathbb{P}^3}(k-d)) = 0$ (see [Har77, Theorem 5.1]). Now, \mathcal{I}_C is $\deg(C)$ -regular (see [Gia06, Main Theorem]). So, $H^1(\mathcal{I}_C(k)) = 0$ for $k \geq \deg(C)$. This implies $H^1(\mathcal{O}_X(-C)(k)) = 0$ for $k \geq \deg(C)$. By Serre duality, $0 = H^1(\mathcal{O}_X(-C)(d-4)) \cong H^1(\mathcal{O}_X(C))$. So, C is semi-regular. \square

Recall the following result,

Lemma 5.3. Let $d \geq 5$ and C be an effective divisor on a smooth degree d surface X of the form $\sum_i a_i C_i$ where C_i are integral curves with $\deg(C) + 2 \leq d$. Then, $h^0(\mathcal{N}_{C|X}) = 0$. In particular, $\dim |C| = 0$ where $|C|$ is the linear system associated to C .

Proof. See [Dan14, Lemma 3.6]. \square

Lemma 5.4. Let X be a smooth surface and C a local complete intersection in X . Then, $h^0(\mathcal{O}_X(-C)(d)) = h^0(\mathcal{I}_C(d)) - 1$ and $h^0(\mathcal{N}_{C|X}) = h^0(\mathcal{O}_X(C)) - 1$.

Proof. The first equality follows from the short exact sequence,

$$0 \rightarrow \mathcal{I}_X(d) \rightarrow \mathcal{I}_C(d) \rightarrow i_* \mathcal{O}_X(-C)(d) \rightarrow 0$$

and the fact that $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}$, where $i : X \rightarrow \mathbb{P}^3$ is the natural closed immersion.

The second equality follows from the short exact sequence (3) after using the facts $h^0(\mathcal{O}_X) = 1$ and $h^1(\mathcal{O}_X) = 0$. \square

We now use these three lemmas given above to produce an explicit formula for the tangent space to the Hodge locus corresponding to the cohomology class of a reduced connected curve.

Proposition 5.5. Let X be a smooth degree d surface in \mathbb{P}^3 and $C \subset X$, a reduced, connected curve satisfying $\deg(X) \geq \deg(C) + 4$. Then

$$\dim T_X(\text{NL}([C])) = h^0(\mathcal{I}_C(d)) - 1 + h^0(\mathcal{N}_{C|\mathbb{P}^3}).$$

Proof. Lemma 5.2 tells us that if $\deg(X) \geq \deg(C) + 4$ and C reduced, connected then $h^1(\mathcal{O}_X(-C)(d)) = 0$ and $h^1(\mathcal{O}_X(C)) = 0$. Corollary 4.10 implies that ρ_C is surjective. Lemma 5.3 implies that $h^0(\mathcal{N}_{C|X}) = 0$, so β_C is injective. Using Lemma 5.4 and Corollary 4.9 we can finally conclude that,

$$\dim T_X(\text{NL}([C])) = h^0(\mathcal{O}_X(-C)(d)) + h^0(\mathcal{N}_{C|\mathbb{P}^3}) - h^0(\mathcal{N}_{C|X}) = h^0(\mathcal{I}_C(d)) - 1 + h^0(\mathcal{N}_{C|\mathbb{P}^3}).$$

This proves the proposition. \square

We recall a result due to Bloch which gives a relation between the Hodge locus corresponding to the cohomology class of a semi-regular curve and the corresponding flag Hilbert scheme.

Theorem 5.6 ([Blo72, Theorem 7.1]). Let $X \rightarrow S$ be a smooth projective morphism with $S = \text{Spec}(\mathbb{C}[[t_1, \dots, t_r]])$. Let $X_0 \subset X$ be the closed fiber and let $Z_0 \subset X_0$ be a local complete intersection subscheme of codimension 1. Suppose that the topological cycle class $[Z_0] \in H^2(X_0, \mathbb{Z})$

lifts to a horizontal class $z \in F^1 H^2(X)$ (where F^\bullet refers to the Hodge filtration) and that Z_0 is semi-regular in X_0 . Then, Z_0 lifts to a subscheme $Z \subset X$.

This result implies the following:

Theorem 5.7. Let X be a surface, C be a semi-regular curve in X and $\gamma \in H^{1,1}(X, \mathbb{Z})$ be the class of C . For any irreducible component L' of $\overline{\text{NL}(\gamma)}$ (the closure is taken in the Zariski topology on U_d) there exists an irreducible component H' of $H_{P, Q_{d_{\text{red}}}}$ containing the pair (C, X) such that $\text{pr}_2(H')$ coincides with L'_{red} , the associated reduced subscheme, where pr_2 is the second projection map from H_{P, Q_d} to H_{Q_d} .

Proof. Using basic deformation theory, one can check that the image under $\text{pr}_2 : H_{P, Q_d} \rightarrow H_{Q_d}$ of all the irreducible components of $H_{P, Q_{d_{\text{red}}}}$ containing the pair (C, X) is contained in $\overline{\text{NL}(\gamma)}$. But, Theorem 5.6 proves the converse i.e., $\overline{\text{NL}(\gamma)}_{\text{red}}$ is contained in $\text{pr}_2(H_{P, Q_d})$. This proves the theorem. \square

The following result gives a geometric classification of the components of the Noether-Lefschetz locus:

Proposition 5.8. Consider an irreducible component L of the Noether-Lefschetz locus. Then, for a general element $X \in L$, there exists $\gamma \in H^{1,1}(X, \mathbb{Z})$ satisfying: γ is the cohomology class of a semi-regular curve, say C , and $\overline{\text{NL}(\gamma)} \cong L$.

Proof. By definition, L is locally a Hodge locus. So, there exists $\gamma \in H^{1,1}(X, \mathbb{Z})$ such that L is locally of the form $\text{NL}(\gamma)$. Consider γ as an element in the Neron-Severi group. Since $\text{NL}(\gamma) \cong \text{NL}(a\gamma)$ for any integer a , we can assume that γ is of the form $\sum_i a_i [C_i]$ where $a_i \in \mathbb{Z}$ and C_i are irreducible curves and $\gamma \cdot H_X > 0$ (otherwise replace γ by $-\gamma$). Since $H_X^2 = d$, $(\gamma + nH_X)^2 > 0$ for $n \gg 0$. Using [Har77, Corollary V.1.8], there exists a local complete intersection curve C in X such that C is linearly equivalent to $n'(\gamma + nH_X)$ for $n' \gg 0$. Since $\text{NL}(\gamma) = \text{NL}(a\gamma + bH_X)$ for any a, b non-zero, we have $\text{NL}(\gamma) = \text{NL}([C])$.

Now, Serre's vanishing theorem and Serre duality implies that for $m \gg 0$, $H^1(\mathcal{O}_X(C)(m)) = 0$. For such m , take $C' \in |C + mH_X|$ and γ to be the cohomology class of C' . Then, $\overline{\text{NL}(\gamma)}$ is isomorphic to L and C' is semi-regular. \square

Corollary 5.9. Let L be an irreducible component of the Noether-Lefschetz locus. Then, for a general element $X \in L$, there exists $\gamma \in H^{1,1}(X, \mathbb{Z})$ satisfying: γ is the cohomology class of a semi-regular curve, say C , $H^1(\mathcal{O}_X(-C)(d)) = 0$ and $\overline{\text{NL}}(\gamma) \cong L$.

Remark 5.10. Before we prove this statement, we point out that the main difference between the above corollary and Proposition 5.8 is that we have the additional outcome that $H^1(\mathcal{O}_X(-C)(d)) = 0$. If one looks at the diagrams in 3.2 and Theorem 4.8 one can see this condition precisely tells us that for every infinitesimal deformation of C , there exists a corresponding infinitesimal deformation of X containing it.

Proof of Corollary 5.9. By Proposition 5.8, there exists a semi-regular curve C and a surface X containing it such that $\overline{\text{NL}}([C])$ is isomorphic to L , where $[C] \in H^{1,1}(X, \mathbb{Z})$ is the cohomology class of C . A lemma of Enriques-Severi-Zariski [Har77, Corollary 7.8] states that for $m \gg 0$, $H^1(\mathcal{O}_X(-C)(d-m)) = 0$, where d is the degree of X . Replacing C by a general element in the linear system, $|C + mH_X|$, we get the corollary. \square

Notation 5.11. Let L be an irreducible component of the Noether-Lefschetz locus and C be as in Proposition 5.8. Denote by P the Hilbert polynomial of C . Since C is semi-regular, Theorem 5.7 implies that there exists an irreducible component of H_{P,Q_d} , say H_γ such that $H_{\gamma_{\text{red}}}$ is an irreducible component of $H_{P,Q_{d_{\text{red}}}}$ and $\text{pr}_2(H_\gamma)_{\text{red}}$ is isomorphic to $L_{\text{red}} \cong \overline{\text{NL}}(\gamma)_{\text{red}}$, where L is as in the previous proposition. Denote by $L_\gamma := \text{pr}_1(H_\gamma)$ and by $T_C L_\gamma := \text{pr}_1(T_{(C,X)} H_\gamma)$. We say that L_γ is *non-reduced* if $\dim L_\gamma < \dim T_C L_\gamma$.

We can then compute the dimension of $\overline{\text{NL}}(\gamma)$ as follows:

Proposition 5.12. Notations as in 5.11. The dimension of $\overline{\text{NL}}(\gamma)$,

$$\dim \overline{\text{NL}}(\gamma) = \dim I_d(C) + \dim L_\gamma - h^0(\mathcal{O}_X(C)),$$

for a generic curve C in L_γ .

Proof. This follows from the fiber dimension theorem (see [Har77, II Ex. 3.22]) which states that for a morphism of finite type between two integral schemes, $f : X \rightarrow Y$, $\dim X = \dim f^{-1}(y) +$

$\dim Y$ for a general point $y \in Y$. We then have the following maps,

$$\mathrm{pr}_1 : H_\gamma \twoheadrightarrow L_\gamma \quad \mathrm{pr}_2 : H_\gamma \twoheadrightarrow \overline{\mathrm{NL}(\gamma)}$$

where the generic fiber of pr_1 is $\mathbb{P}(I_d(C))$ and that of pr_2 is $\mathbb{P}(H^0(\mathcal{O}_X(C)))$. We then conclude,

$$\dim H_\gamma = \dim L_\gamma + \dim I_d(C) - 1 = \dim \overline{\mathrm{NL}(\gamma)} + h^0(\mathcal{O}_X(C)) - 1.$$

Therefore, $\dim \overline{\mathrm{NL}(\gamma)} = \dim L_\gamma + \dim I_d(C) - h^0(\mathcal{O}_X(C))$. \square

We finally come to the main result of the section which tells us of a necessary and sufficient criterion for non-reducedness of irreducible components of NL_d :

Theorem 5.13 (Non-reducedness). Let L be an irreducible component of the Noether-Lefschetz locus and C be a semi-regular curve such that $L \cong \overline{\mathrm{NL}([C])}$. Then, L is non-reduced if and only if L_γ is non-reduced in the sense of 5.11. In particular, L is non-reduced if and only if H_γ is non-reduced, where H_γ is as constructed in 5.11.

Proof. Consider C general in L_γ . Using the commutative diagram in Theomre 4.8 we have that

$$\mathrm{Im} \rho_C \cap \mathrm{Im} \beta_C = \rho_C \circ \mathrm{pr}_2(T_{(C,X)} H_{P,Q_d}) = \beta_C \circ \mathrm{pr}_1(T_{(C,X)} H_{P,Q_d}).$$

By Corollary 4.9, $T_X(\mathrm{NL}(\gamma)) = \mathrm{pr}_2(T_{(C,X)} H_{P,Q_d})$. Therefore, Corollary 4.10 implies that

$$\begin{aligned} \dim T_X \mathrm{NL}(\gamma) &= h^0(\mathcal{O}_X(-C)(d)) + \dim T_C L_\gamma - h^0(\mathcal{N}_{C|X}) \\ &= \dim I_d(C) + \dim T_C L_\gamma - h^0(\mathcal{O}_X(C)) \end{aligned}$$

where the last equality follows from Lemma 5.4.

Lemma 5.12 implies that

$$\dim T_X \mathrm{NL}(\gamma) - \dim \mathrm{NL}(\gamma) = \dim T_C(L_\gamma) - \dim L_\gamma.$$

So, $\mathrm{NL}(\gamma)$ is non-reduced if and only if L_γ is non-reduced.

Furthermore, using 3.2 and the fact that the fiber over $C \in L_\gamma$ to the surjective projection morphism from H_γ to L_γ is $\mathbb{P}(I_d(C))$ one can see that

$$\dim T_{(C,X)}H_\gamma = \dim T_C L_\gamma + h^0(\mathcal{O}_X(-C)(d)), \quad \dim H_\gamma = \dim L_\gamma + \dim \mathbb{P}(I_d(C)),$$

where $I_d(C)$ is the degree d graded piece in the ideal of C . Since $h^0(\mathcal{O}_X(-C)(d)) = \dim \mathbb{P}(I_d(C))$, H_γ is non-reduced if and only if $T_C L_\gamma$ is non-reduced. This proves the theorem. \square

6 Examples

In this section we first look at some examples where a component of the Hilbert scheme is non-reduced. We then use these results in §6.4 to give examples of a few non-reduced components of the Noether-Lefschetz locus.

6.1 A smooth example

Theorem 6.1 (Kleppe [Kle85]). There exists an irreducible component L of the Hilbert scheme of curves in \mathbb{P}^3 such that a general curve C in L is smooth, contained in a cubic surface and $h^1(\mathcal{I}_C(3)) \neq 0$ for the following range:

1. If $\deg(C) \geq 18$ then $g(C) > 7 + (\deg(C) - 2)^2/8$.
2. and $g(C) > -1 + (\deg(C)^2 - 4)/8$ for $17 \geq \deg(C) \geq 14$

Remark 6.2. This is a generalization of an example due to Mumford. See [Mum62] or [Har10, §13] for further details on his example.

6.2 Martin-Deschamps and Perrin's example

Notation 6.3. Let a, d be positive integers, $d \geq 5$ and $a > 0$. Let X be a smooth projective surface in \mathbb{P}^3 of degree d containing a line l and a smooth coplanar curve C_1 of degree a . Let C be a divisor of the form $2l + C_1$ in X . Denote by P the Hilbert polynomial of C .

Theorem 6.4 (Martin-Deschamps and Perrin [MDP96]). There exists an irreducible component, say L of H_P such that a general curve $D \in L$ is a divisor in a smooth degree d surface in \mathbb{P}^3 of the

form $2l' + C'_1$ where l', C'_1 are coplanar curves with $\deg(l') = 1$ and $\deg(C'_1) = a$. Furthermore, L is non-reduced.

Proof. The theorem follows from [MDP96, Proposition 0.6, Theorems 2.4, 3.1]. \square

6.3 Computing Castelnuovo-Mumford regularity

6.5. In this section, we show that the Castelnuovo-Mumford regularity of the curves considered in Notation 6.3 is equal to d .

Notation 6.6. Denote by S the ring $\Gamma_*(\mathcal{O}_{\mathbb{P}^3}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n))$. Let C_0 be a plane curve of degree $a + 1$ for some positive integer a , defined by two equations, say $l_1, l_2 G_1$ where l_1, l_2 are linear polynomials and G_1 is a smooth polynomial of degree a . Let X be a smooth surface of degree d containing C_0 . Then X is defined by an equation of the form $F_X := F_1 l_1 + F_2 l_2 G_1$. By taking X to be a general surface containing C_0 , we can assume that G_1, F_1 and F_2 are not contained in the ideal generated by l_1, l_2 . Denote by l the line defined by l_1 and l_2 and by A_l its coordinate ring. The aim of this section is to prove that $\mathcal{O}_X(-l) \otimes \mathcal{O}_X(-C_0)$ is d -regular.

We now recall a result on Castelnuovo-Mumford regularity which we will use later:

Theorem 6.7 ([Eis05, Ex. 4E, Proposition 4.16]). Suppose that

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is a short exact sequence of finitely generated S -modules. Denote by $\text{reg}(N)$ the Castelnuovo-Mumford regularity of a finitely generated S -module N . Then, $\text{reg}(M') \leq \max\{\text{reg}(M), \text{reg}(M'') - 1\}$, $\text{reg}(M'') \leq \max\{\text{reg}(M), \text{reg}(M') + 1\}$, $\text{reg}(M) \leq \max\{\text{reg}(M'), \text{reg}(M'')\}$. Furthermore, given a finitely generated S -module M , $\text{reg}(M) \geq \text{reg}(\widetilde{M})$ where \widetilde{M} is the associated coherent sheaf.

6.8. Denote by I_{C_0} (resp. I_X, I_l) the ideal of C_0 (resp. X, l). Then we have the following

commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & I_X & \xrightarrow{f_1} & I_{C_0} & \xrightarrow{f_2} & I_{C_0}/I_X \longrightarrow 0 \\
& & \uparrow \cong & \circlearrowleft & \uparrow f_4 & & \\
& & S(-d) & \xrightarrow{f_3} & S(-1) \oplus S(-a-1) & & \\
& & & & \uparrow f_5 & & \\
& & & & S(-2-a) & &
\end{array} \tag{9}$$

where f_1 is the natural inclusion map, f_2 is the quotient map,

$$f_3 : G \mapsto (GF_1, GF_2), \quad f_4 : (H_1, H_2) \mapsto H_1 l_1 + H_2 l_2 G_1, \quad f_5 : G \mapsto (Gl_2 G_1, -Gl_1).$$

Note that the top horizontal row and the rightmost vertical sequence are exact.

Lemma 6.9. The kernel of the natural morphism $f'_5 + f'_3$ from $A_l(-2-a) \oplus A_l(-d)$ to $A_l(-1) \oplus A_l(-a-1)$ which maps $(\overline{G}, \overline{H})$ to $(\overline{HF_1 + Gl_2 G_1}, \overline{HF_2 - Gl_1})$ is $A_l(-a-2) \oplus 0$, where $G, H \in S(-a-1), S(-d)$, respectively and $\overline{G}, \overline{H}$ are its images in $A_l(-2-a)$ and $A_l(-d)$, respectively

Proof. Since I_l is generated by l_1 and l_2 , $\overline{HF_1 + Gl_2 G_1}$ and $\overline{HF_2 - Gl_1}$ are both zero if and only if both HF_1 and HF_2 are both in I_l . Since F_1, F_2 are not contained in I_l , by assumption, and I_l is a prime ideal, this implies H is contained in I_l . In other words, the kernel of the map $f'_5 + f'_3$ is isomorphic to pairs $(\overline{G}, 0)$ where $G \in S(-a-2)$. This proves the lemma. \square

Proposition 6.10. $A_l \otimes_S I_{C_0}/I_X$ is $d+1$ -regular.

Proof. Using the exactness of the sequences in the diagram (9), we can conclude that $\text{Im } f_3 + \text{Im } f_5$ is contained in $\ker(f_2 \circ f_4)$. We now show the converse inclusion. Let $h \in \ker(f_2 \circ f_4)$. So, $f_4(h) \in \text{Im } f_1$, which implies that there exists $b \in S(-d)$ such that $f_4(h - f_3(b)) = 0$. Therefore, there exists $c \in S(-2-a)$ such that $f_5(c) = h - f_3(b)$. This proves that $\ker(f_2 \circ f_4) \subset \text{Im } f_3 + \text{Im } f_5$, hence equality.

Note that the image of f_5 maps to zero under f_4 due to exactness of the sequence. But, $f_4 \circ f_3$ is injective by the commutative of the above diagram and the injectivity of f_1 . So,

$\text{Im } f_3 \cap \text{Im } f_5 = 0$. So, $\ker(f_4 \circ f_2) = S(2 - a) \oplus S(-d)$. So, we have the following short exact sequence of S -modules:

$$0 \rightarrow S(-2 - a) \oplus S(-d) \xrightarrow{f_5 \oplus f_3} S(-1) \oplus S(-a - 1) \xrightarrow{f_4 \circ f_2} I_{C_0}/I_X \rightarrow 0 \quad (10)$$

Since tensor product is right exact, using Lemma 6.9 we get the following exact sequence:

$$0 \rightarrow A_l(-d) \xrightarrow{f'_3} A_l(-1) \oplus A_l(-a - 1) \xrightarrow{f'_4 \circ f'_2} A_l \otimes_S I_{C_0}/I_X \rightarrow 0$$

where f'_i are induced by f_i for $i = 1, \dots, 5$.

Using the short exact sequence

$$0 \rightarrow S(-2) \rightarrow S(-1) \oplus S(-1) \rightarrow A_l \rightarrow 0$$

we see that the Castelnuovo-Mumford regularity of $A_l(-d)$ and $A_l(-1) \oplus A_l(-a - 1)$ is less than or equal to d . Finally, Theorem 6.7 implies that the Castelnuovo-Mumford regularity of $A_l \otimes_S I_{C_0}/I_X$ is at most $d + 1$. \square

Theorem 6.11. The sheaf $\mathcal{O}_X(-l - C_0)$ is d -regular.

Proof. Consider the natural surjective morphism $A_X \rightarrow A_l$. Tensoring this by I_{C_0}/I_X , we obtain the short exact sequence,

$$0 \rightarrow \ker(p) \rightarrow I_{C_0}/I_X \xrightarrow{p} A_l \otimes_S I_{C_0}/I_X \rightarrow 0.$$

Using Proposition 6.10 and Theorem 6.7 we conclude that $\widetilde{\ker(p)}$ is d -regular. It remains to prove that $\widetilde{\ker p} \cong \mathcal{O}_X(-C_0 - l)$.

Since $\Gamma_*(\mathcal{O}_X(-C_0)) = I_{C_0}/I_X$ (by definition) and $\Gamma_* \mathcal{O}_l = A_l$, [Har77, Proposition II.5.15] implies that $\widetilde{I_{C_0}/I_X}$ and $\widetilde{A_l}$ is isomorphic to $\mathcal{O}_X(-C_0)$ and \mathcal{O}_l , respectively. Now, the associated module functor $\widetilde{}$ is exact and commutes with tensor product ([Har77, Proposition II.5.2]). Applying this functor to the last short exact sequence we get,

$$0 \rightarrow \widetilde{\ker(p)} \rightarrow \mathcal{O}_X(-C_0) \xrightarrow{\widetilde{p}} \mathcal{O}_l \otimes \mathcal{O}_X(-C_0) \rightarrow 0$$

where \tilde{p} arises from tensoring by $\mathcal{O}_X(-C_0)$ the natural surjective morphism $\mathcal{O}_X \rightarrow \mathcal{O}_l$.

Consider now the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-l) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_l \rightarrow 0.$$

Since $\mathcal{O}_X(-C_0)$ is a flat \mathcal{O}_X -module, we get the short exact sequence,

$$0 \rightarrow \mathcal{O}_X(-C_0 - l) \rightarrow \mathcal{O}_X(-C_0) \xrightarrow{\tilde{p}} \mathcal{O}_l \otimes_{\mathcal{O}_X} \mathcal{O}_X(-C_0) \rightarrow 0.$$

By the universal property of the kernel, $\widetilde{\ker p}$ is isomorphic to $\mathcal{O}_X(-C_0 - l)$. Hence, $\mathcal{O}_X(-C_0 - l)$ is d -regular. \square

6.4 Examples of Non-reduced components of NL_d

6.12. Before we come to the final result of this article we recall a result by Kleiman and Altman which tells us given a curve when does there exist a *smooth* surface in \mathbb{P}^3 containing it. This will be used to prove the existence of certain components of the Noether-Lefschetz locus. We then prove non-reducedness.

Notation 6.13. Let C be a projective curve in \mathbb{P}^3 . Denote by

$$D_e := \{x \in C \mid \dim \Omega_{C,x}^1 = e\}.$$

The theorem in this case states that,

Theorem 6.14 ([KA79, Theorem 7]). If for any $e > 0$ such that $D_e \neq \emptyset$ we have that $\dim D_e + e$ is less than 3 then there exists a smooth surface in \mathbb{P}^3 containing C . Moreover, if C is $d-1$ -regular then there exists a smooth degree d surface containing C .

6.15. We now collect the previous results to give some examples of non-reduced components of NL_d . In the following theorem the phrase *C is general* would mean that the point on the Hilbert scheme corresponding to the curve C is a smooth point which implies that no further irreducible component of the Hilbert scheme pass through this point.

Theorem 6.16. The following statements are true:

1. Let C be a smooth curve or a curve with at most double points (i.e., points $x \in C_{\text{red}}$ such that $\dim \Omega_{C_{\text{red}},x}^1 = 2$). Take $d > \deg(C) + 4$. Then there exists smooth degree d surfaces in \mathbb{P}^3 containing C .
2. Let C be a curve and $C \subset X$, a degree d surface. Let $\gamma = [C]$. Then $\text{NL}(\gamma)$ is non-reduced when
 - (a) C is a general element in L as in Theorem 6.1 and $d \geq \deg(C) + 4$.
 - (b) C is a general element in L as in Theorem 6.4 and $\overline{\text{NL}(\gamma)}_{\text{red}}$ coincides with $\text{pr}_2(\text{pr}_1^{-1}(L))_{\text{red}}$ where P is the Hilbert polynomial of C and pr_i are natural projection morphisms from H_{P,Q_d} to the respective components.

Proof. 1. It follows from the proof of Lemma 5.2 that C_{red} is $d - 1$ regular. Using Theorem 6.14, we can then conclude that there exists smooth degree d surfaces containing C .

2. Note that for all C as in (a) and (b), C is either smooth or C_{red} has at most double points. Using (1) there exists smooth surfaces X in \mathbb{P}^3 containing such C of degree d . Take $\gamma = [C]$. We now prove non-reducedness.

- (a) Lemma 5.2 tells us that C is semi-regular and $h^1(\mathcal{O}_X(-C)(d)) = 0$. Denote by P the Hilbert polynomial of C . Theorem 5.7 implies that there exists an irreducible component H_γ in H_{P,Q_d} such that $\text{pr}_2(H_\gamma)_{\text{red}} = \overline{\text{NL}(\gamma)}_{\text{red}}$. Since C is general in L , $\text{pr}_1(H_\gamma)_{\text{red}} = L_{\text{red}}$. Using Corollary 4.10 we notice that the natural morphism from $H^0(\mathcal{N}_{X|\mathbb{P}^3})$ to $H^0(\mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C)$ is surjective and $T_C L_\gamma = H^0(\mathcal{N}_{C|\mathbb{P}^3})$, where $L_\gamma = \text{pr}_1(H_\gamma)$ and $T_C L_\gamma$ is as defined in 5.11. Theorem 6.1 states that $\dim L_\gamma < h^0(\mathcal{N}_{C|\mathbb{P}^3})$ for C in L_γ , general. Finally, Theorem 5.13 shows that $\overline{\text{NL}(\gamma)}$ is non-reduced.
- (b) Using Theorem 6.11, we have that $h^1(\mathcal{O}_X(-C)(d)) = 0$, hence Corollary 4.10 implies the natural morphism from $H^0(\mathcal{N}_{X|\mathbb{P}^3})$ to $H^0(\mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C)$ is surjective. Following exactly as in the proof of Theorem 5.12 by replacing H_γ by an irreducible component in $\text{pr}_1^{-1}(L)$ such that $\text{pr}_2(H_\gamma)_{\text{red}} \cong \overline{\text{NL}(\gamma)}$, we get after using Lemma 5.3 that $\dim \text{NL}(\gamma) = \dim I_d(C) - 1 + \dim L$. Using Corollary 4.9 and Lemma 5.3 we get $\dim T_X(\text{NL}(\gamma)) \geq$

$\dim I_d(C) - 1 + h^0(\mathcal{N}_{C|\mathbb{P}^3})$. Theorem 6.4 implies $\dim L < h^0(\mathcal{N}_{C|\mathbb{P}^3})$, hence non-reducedness of $\overline{\text{NL}(\gamma)}$.

□

We give an example satisfying the conditions in Theorem 6.162(b).

Theorem 6.17. Let $d \geq 5$, X be a smooth degree d surface containing 2 coplanar lines, l_1, l_2 . Let C be a divisor in X of the form $2l_1 + l_2$, γ the cohomology class of C . Then, $\overline{\text{NL}(\gamma)}_{\text{red}}$ is isomorphic to $\overline{(\text{NL}([l_1]) \cap \text{NL}([l_2]))}_{\text{red}}$. In particular, $\overline{\text{NL}(\gamma)}$ is non-reduced.

Proof. Note that $\text{NL}(\gamma)$ contains the space of smooth degree d surfaces containing two coplanar lines. But this space is of codimension $2d-6$. So, $\text{codim } \overline{\text{NL}(\gamma)} \leq 2d-6$. If $\text{codim } \overline{\text{NL}(\gamma)} < 2d-6$ [Voi89, Proposition 1.1] implies that $\overline{\text{NL}(\gamma)}$ is reduced and either it parametrizes smooth degree d surfaces containing a line or containing a conic. This means that there exists γ' a class of a line or a conic in X such that $\text{NL}(\gamma) = \text{NL}(\gamma')$. Note that if γ'_{prim} is a multiple of $[l_1]_{\text{prim}}$ or $[l_2]_{\text{prim}}$ or $[l_1 + l_2]_{\text{prim}}$ then this locus parametrizes surfaces such that the cohomology class of l_1 and l_2 remains of type $(1, 1)$. Lemma 5.2 tells us l_1, l_2 are semi-regular for $d \geq 5$. Theorem 5.7 then implies this space parametrizes surfaces containing 2 coplanar lines, hence $\text{codim } \overline{\text{NL}(\gamma)} = 2d-6$, which gives us a contradiction.

Choose X in $\text{NL}(\gamma)$ such that if H is the hyperplane containing $l_1 \cup l_2$ then $H \cap X = l_1 \cup l_2 \cup D$ with D irreducible. Note that such X exists and the new γ corresponding to the cohomology class of the divisor $2l_1 + l_2$ in this surface defines the same component $\overline{\text{NL}(\gamma)}$. Denote by E the preimage in $H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ of the vector space $T_X(\text{NL}(\gamma)) \subset H^0(\mathcal{O}_X(d))$. For an integer k , denote by

$$E_k := [E : S_{d-k}] := \{P \in S_k | P.S_{d-k} \subset E\},$$

where S_{d-k} is $H^0(\mathcal{O}_{\mathbb{P}^3}(d-k))$. Now, [GH83, 4.a.5] implies that $I(l_1 \cup l_2)$ is contained in $\oplus_k E_k$. Now, [Mac05, Proposition 4] and the argument after [Mac05, Proposition 5] tells us that $\gamma' = a_1[l_1] + a_2[l_2] + b[H_X]$, where γ' is the class $[C']$ of a line or a conic and a_1, a_2, a_3 are rationals.

By the above argument, C' is not l_1, l_2 or $l_1 \cup l_2$. Denote by t_0 and t_1 the integers $l_1.C'$ and $l_2.C'$, respectively. Then, intersecting the equality $[C'] = a_1[l_1] + a_2[l_2] + b[H_X]$ by H_X, l_1, l_2 and

C' , respectively, we have

$$\deg(C') = a_1 + a_2 + bd \quad (11)$$

$$t_0 = a_1(2 - d) + a_2 + b \quad (12)$$

$$t_1 = a_1 + a_2(2 - d) + b \quad (13)$$

$$C'^2 = a_1 t_0 + a_2 t_1 + \deg(C')b \quad (14)$$

Assume that $l_i \not\subset C'$. Using adjunction formula one can check C'^2 is $2 - d$ if C' is a line and $6 - 2d$ if C' is a conic. Then, using any standard programming language (for eg. Maple), one can see that there does not exist a solution to these set of equations.

The only case that remains is when C' is a conic and of the form $l' \cup l_i$ for $i = 1$ or 2 and l' is distinct from l_1, l_2 . We can then replace in the above equation C' by l' and replace a_1 (resp. a_2) by $a_1 - 1$ (resp. $a_2 - 1$) if $i = 1$ (resp. $i = 2$). Then the above result tells us again that there are no solutions. So, $\text{codim } \overline{\text{NL}}(\gamma)$ has to be $2d - 6$. \square

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